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Reconsidering the existence of stable solutions in three-sided matching problems with mixed preferences

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Abstract
We reexamine the existence of stable solutions in a class of three-sided matching problems previously studied by Zhang and Zhong (J Comb Optim 42:928–245, 2021). The sets of participants are U, V, and W. Agents in U have strict preferences defined on V, agents in V have strict preferences defined on W, and agents in W have strict preferences defined on $U \times V$. In this framework, we show that a *weakly stable matching* may not exist.

Keywords Matching markets · Stability · Mixed preferences

JEL Classification D47 · C78

1 Introduction

In any corporation, the board of directors cares about managers' profiles, who in turn are concerned about the attributes of team leaders. When the latter have preferences for the characteristics of their managers and directors, it is natural to ask whether a stable distribution of executives in committees that include a director, a manager, and a team leader exists. In this work, we address this question from the perspective of matching theory, arriving at a negative answer even in scenarios where team leaders are primarily concerned about directors' attributes.

More precisely, we analyze the existence of stable outcomes in a matching problem with three sides, U, V and W, and where agents in U have strict preferences defined on V, agents in V have strict preferences defined on W, and agents in W have strict

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preferences defined on $U \times V$. Zhang and Zhong (2021) referred to this problem as a *three-sided matching problem with mixed preferences*.[1]

We focus on the existence of matchings—distributions of agents in triplets of $U \times V \times W$—that are *weakly stable* or *stable*. A matching is weakly stable when no group of three agents of different market sides may form a new triplet to improve the well-being of *all* of its members. A matching is stable when no group of three agents of different market sides may form a new triplet to improve the well-being of those who change their relevant partners. Any stable matching is weakly stable.

Zhang and Zhong (2021) proposed a constructive way to find a stable matching in a three-sided matching problem with mixed preferences. In our first result, and by means of an example, we show that the algorithm proposed by these authors does not always lead to a weakly stable matching.

More importantly, we assert that weakly stable matchings do not always exist in this environment. To show this result, we rely on the findings of Lam and Plaxton (2021), which guarantee the non-existence of weakly stable outcomes in some instances of the classical *three-sided matching problem with cyclic preferences*, where agents in W have strict preferences defined on U instead of $U \times V$.

The rest of the paper is organized as follows. Section 2 describes the characteristics of the three-sided matching problem with mixed preferences and the stability concepts that will be studied. Section 3 provides an example that shows that the algorithm proposed by Zhang and Zhong (2021) may fail to generate a (weakly) stable outcome. Section 4 shows that a three-sided matching problem with mixed preferences may have an empty set of weakly stable matchings.

2 The model

A *three-sided matching problem with mixed preferences*, represented by $[U, V, W, (\succ_h)_{h\in H}]$, is characterized by a set of agents $H = U \cup V \cup W$ and a preference profile $(\succ_h)_{h\in H}$ such that:

- U, V, and W are disjoint sets satisfying $|U| = |V| = |W|$.
- $(\succ_u)_{u\in U}$ are strict linear orders defined on V.
- $(\succ_v)_{v\in V}$ are strict linear orders defined on W.
- $(\succ_w)_{w\in W}$ are strict linear orders defined on $U \times V$.

In this context, a *matching* is a set $M \subseteq U \times V \times W$ such that any $h \in H$ belongs to one and only one triplet in M. If a triplet (u, v, w) belongs to a matching M, then the *relevant partners* of each of its members are denoted by $M(u) = v$, $M(v) = w$, and $M(w) = (u, v)$.

Definition 1 A matching M is *strongly blocked* by a triplet $(u, v, w) \in U \times V \times W$ when $(u, v, w) \notin M$ and we have that $v \succ_u M(u)$, $w \succ_v M(v)$, and $(u, v) \succ_w M(w)$.

[1] Under alternative assumptions on preferences, there is a vast literature that focuses on the existence of stable outcomes in three-sided matching problems (see, for instance, Danilov 2003; Boros et al. 2004; Eriksson et al. 2006; Biró and McDermid 2010; Huang 2010; Zhang et al. 2019; Zhong and Bai 2019; Pashkovich and Poirrier 2020; Lam and Plaxton 2019, 2021).

A matching is *weakly stable* when it cannot be strongly blocked by any triplet. Thus, a weakly stable matching is a distribution of the population in triplets of $U \times V \times W$ such that no group of three agents of different market sides may form a new triplet to improve the well-being of *all* of its members.

Definition 2 A matching M is *blocked* by a triplet $(u, v, w) \in U \times V \times W$ when $(u, v, w) \notin M$ and the following conditions are satisfied:

- If $v \neq M(u)$, then $v \succ_u M(u)$.
- If $w \neq M(v)$, then $w \succ_v M(v)$.
- $(u, v) \succ_w M(w)$.

A matching is *stable* when it cannot be blocked by any triplet.

In a stable matching, no group of three agents of different market sides may deviate, forming a new triplet to improve the well-being of members who change their relevant partners. Evidently, any stable matching is weakly stable.

3 The algorithm and the counterexample

In a recent work, Zhang and Zhong (2021, Section 4) proposed a constructive way to find a stable matching for the three-sided problem described above. In this section, we show that this process may generate a matching that is not even weakly stable.
The algorithm proposed by these authors is the following:

Step 0. Each agent in U is *active* and M is an empty set. For each $(u, v) \in U \times V$, $\succ_{u,v}$ is a strict linear order that coincides with $\succ_v$.

Step 1. Each *active* $u \in U$, according to the linear order $\succ_u$, proposes to the most preferable *active* agent in V and mark it as v. We get a pair (u, v). The set of pairs obtained in this way is denoted by K.

Step 2. Each pair $(u, v) \in K$, according to the linear order $\succ_{u,v}$, proposes to the most preferable one in W and mark it as w. We get a triplet (u, v, w). All triplets obtained in this way are denoted as Q. Let $R = Q \cup M$, and $R|_W = \{w \in W \mid \exists (u, v) \in U \times V,\ (u, v, w) \in R\}$.

Step 3. Each agent $\bar{w} \in R|_W$, according to the linear order $\succ_{\bar{w}}$, chooses the most preferable one among the pairs proposing to him and mark it as $(\bar{u}, \bar{v})$ (if $M(w) = (u', v')$, then (u', v') is included among these pairs). Any pair $(u, v) \neq (\bar{u}, \bar{v})$ that proposed to $\bar{w}$ is rejected and $\bar{w}$ is *removed* from the linear order $\succ_{u,v}$.

Step 4. If $M(\bar{w}) = \emptyset$, then let $M := M \cup \{(\bar{u}, \bar{v}, \bar{w})\}$, and mark $\bar{u}$ and $\bar{v}$ as *inactive*.
Alternatively, if $M(\bar{w}) = (u', v')$ and $(\bar{u}, \bar{v}) \neq (u', v')$, then let $M = M \cup \{(\bar{u}, \bar{v}, \bar{w})\} \setminus \{(u', v', \bar{w})\}$, mark $\bar{u}$ and $\bar{v}$ as *inactive*, and *reactivate* u' and v'.

Step 5. Repeat Steps 1, 2, 3, and 4 until all agents $u \in U$ are *inactive*.

The following example describes a family of preference profiles for which the algorithm outlined above fails to generate a (weakly) stable matching.

Example 1 Assume that $U = \{u_1, u_2, u_3, u_4\}$, $V = \{v_1, v_2, v_3, v_4\}$, $W = \{w_1, w_2, w_3, w_4\}$, and that preference relations are determined by strict linear orders such that:

$$
\begin{array}{lll}
v_2 \succ_{u_1} v_3 \succ_{u_1} v_1 \succ_{u_1} v_4 & w_1 \succ_{v_1} w_2 \succ_{v_1} \cdots & (u_1, v_1) \succ_{w_1} (u_2, v_2) \succ_{w_1} \cdots \\
v_2 \succ_{u_2} \cdots & w_1 \succ_{v_2} \cdots & (u_3, v_2) \succ_{w_2} \cdots \\
v_1 \succ_{u_3} \cdots & w_3 \succ_{v_3} \cdots & (u_4, v_3) \succ_{w_3} \cdots \\
v_2 \succ_{u_4} v_3 \succ_{u_4} \cdots & w_4 \succ_{v_4} \cdots & (u_2, v_4) \succ_{w_4} \cdots
\end{array}
$$

Applying the algorithm of Zhang and Zhong (2021, Section 4) we obtain that:

(1a) Agents u_1, u_2 and u_4 propose to v_2, while agent u_3 proposes to v_1.
(2a) All pairs in $\{(u_1, v_2), (u_2, v_2), (u_3, v_1), (u_4, v_2)\}$ propose to w_1.
(3a) Agent w_1 chooses (u_2, v_2) and rejects the other pairs. Thus, w_1 is removed from $\succ_{u_1,v_2}$, $\succ_{u_3,v_1}$, and $\succ_{u_4,v_2}$.
(4a) $M = \{(u_2, v_2, w_1)\}$. Agents u_2 and v_2 are marked as inactive.
(1b) Agent u_1 proposes to v_3, agent u_3 proposes to v_1, and agent u_4 proposes to v_3.
(2b) Pairs (u_1, v_3) and (u_4, v_3) propose to w_3. Since w_1 was removed from $\succ_{u_3,v_1}$, the pair (u_3, v_1) proposes to w_2.
(3b) The agent w_3 chooses (u_4, v_3) and rejects (u_1, v_3). Thus, w_3 is removed from $\succ_{u_1,v_3}$. The agent w_2 chooses (u_3, v_1).
(4b) $M = \{(u_2, v_2, w_1), (u_4, v_3, w_3), (u_3, v_1, w_2)\}$. Only agents u_1 and v_4 remain active.
(1c) Agent u_1 proposes to v_4.
(2c) The pair (u_1, v_4) proposes to w_4.
(3c) The agent w_4 chooses (u_1, v_4).
(4c) $M = \{(u_2, v_2, w_1), (u_4, v_3, w_3), (u_3, v_1, w_2), (u_1, v_4, w_4)\}$. All agents in U are inactive.

The matching obtained, $M = \{(u_2, v_2, w_1), (u_4, v_3, w_3), (u_3, v_1, w_2), (u_1, v_4, w_4)\}$, is not weakly stable as it is strongly blocked by the triplet (u_1, v_1, w_1). Indeed, we have that $v_1 \succ_{u_1} v_4$, $w_1 \succ_{v_1} w_2$, and $(u_1, v_1) \succ_{w_1} (u_2, w_2)$.[2] □

Notice that the matching

$$\{(u_1, v_1, w_1), (u_4, v_3, w_3), (u_3, v_2, w_2), (u_2, v_4, w_4)\}$$

is stable in any problem compatible with the requirements of Example 1, because agents in W are paired with their best alternatives in $U \times W$. Hence, one would think that there is a way to ensure that any instance of $[U, V, W, (\succ_h)_{h\in H}]$ has a (weakly) stable matching.

[2] In our example, $(u_1, v_1) \notin M|_{U\times V}$ and $(v_1, w_1) \notin M|_{V\times W}$. Moreover, $(u_1, v_4, w_4) \in M$ and agent u_1 never proposes to v_1. In the proof of Theorem 3 in Zhang and Zhong (2021, page 933) it is argued that these properties imply that $v_4 \succ_{u_1} v_1$ (see the "case (3)" in the proof of this result). Unfortunately, this is not necessarily true, as in Example 1 we have that $v_1 \succ_{u_1} v_4$. Notice that the reason why agent u_1 never proposes to v_1 is that v_1 was *inactive* when u_1 wanted to propose to her.

4 Non-existence of weakly stable matchings

In a recent work, Lam and Plaxton (2021) show that there are instances of the classical *three-sided matching problem with cyclic preferences* that has no weakly stable outcome. In what follows, we exploit their results to show that a weakly stable matching may not exist in some instances of the three-sided matching problem with mixed preferences.

Definition 3 A strict linear order $\succ$ defined on $U \times V$ is *strongly* UV-*lexicographic* when there is a strict linear order $\succ_U$ defined on U and a strict linear order $\succ_V$ defined on V, such that

$$(u, v) \succ (u', v') \quad \Longleftrightarrow \quad [\, u \succ_U u' \,] \quad \text{or} \quad [\, u = u' \ \text{ and } \ v \succ_V v' \,].$$

Lam and Plaxton (2021) proceed in two steps to show that there are instances of the three-sided matching problem with cyclic preferences with no weakly stable outcomes. First, given an instance of a three-sided matching problem with *incomplete linear orders* and cyclic preferences (3MI-CYC from now on), they construct an instance of a three-sided matching problem with *complete linear orders* and cyclic preferences (3MC-CYC from now on). Second, they show that any weakly stable matching in the constructed 3MC-CYC instance induces a weakly stable matching in the original 3MI-CYC instance. Hence, as Biró and McDermid (2010, Lemma 1) show that there are 3MI-CYC instances without weakly stable matchings, they conclude that there are 3MC-CYC instances without weakly stable outcomes.

To show that a weakly stable matching may not exist in some instances of the three-sided matching problem with mixed preferences, our strategy is to appeal to Lam and Plaxton (2021). Essentially, we will reduce the existence of weakly stable matchings in a 3MC-CYC instance to the existence of weakly stable matchings in a three-sided matching problem with mixed preferences in which agents of W have strongly UV-lexicographic preferences.

Theorem 1 *The set of weakly stable matchings of a problem* $[U, V, W, (\succ_h)_{h\in H}]$ *may be empty, even when linear orders* $(\succ_w)_{w\in W}$ *are strongly* UV-*lexicographic.*

Proof Let $[U, V, W, (\succ_h)_{h\in H}]$ be a problem where $(\succ_w)_{w\in W}$ are strongly UV-lexicographic linear orders. Hence, for each $w \in W$, there are strict linear orders $\succ_{U,w}$ and $\succ_{V,w}$ such that

$$(u, v) \succ_w (u', v') \quad \Longleftrightarrow \quad [\, u \succ_{U,w} u' \,] \quad \text{or} \quad [\, u = u' \ \text{ and } \ v \succ_{V,w} v' \,].$$

Notice that, any matching that is weakly stable in $[U, V, W, (\succ_h)_{h\in H}]$ is also weakly stable in the three-sided matching problem with *ciclic preferences* $[U, V, W, (\succ_h)_{h\in U\cup V}, (\succ_{U,w})_{w\in W}]$. Since Lam and Plaxton (2021, Theorem 4) show that there are three-sided problems with cyclic preferences without weakly stable matchings, we conclude that there are problems $[U, V, W, (\succ_h)_{h\in H}]$ that have no weakly stable matchings.[3] □

[3] The 3MC-CYC instance with no weakly stable matchings (implicitly) constructed by Lam and Plaxton (2019, 2021) has 90 agents on each market side. Recently, Lerner (2022) provides an example of a 3MC-

In light of Theorem 1, we can reexamine the relationship between two types of matching problems previously studied by Zhang and Zhong (2021). Given a three-sided matching problem with mixed preferences $[U, V, W, (\succ_h)_{h\in H}]$, suppose that linear orders $(\succ_w)_{w\in W}$ are strongly UV-lexicographic and denote by $[U, V, W, (\succ_h)_{h\in U\cup V}, (\succ_{U,w}, \succ_{V,w})_{w\in W}]$ the three-sided matching problem in which agents in W have *separable preferences* for U and W, where $\succ_{U,w}$ and $\succ_{V,w}$ are the strict linear orders underlying $\succ_w$. In the latter problem, a matching M is *stable* when there is no triplet $(u, v, w) \in U \times V \times W$ such that $(u, v, w) \notin M$ and the following conditions hold:

- If $v \neq M(u)$, then $v \succ_u M(u)$.
- If $w \neq M(v)$, then $w \succ_v M(v)$ and $v \succ_{V,w} M_V(w)$.
- If $u \neq M_U(w)$, then $u \succ_{U,w} M_U(w)$.

where $(M_U(w), M_V(w)) \in U \times V$ is the pair of agents that form a triplet with w in M.[4] In this context, Zhang and Zhong (2021, Theorem 4) claim that any stable matching of the problem $[U, V, W, (\succ_h)_{h\in U\cup V}, (\succ_{U,w}, \succ_{V,w})_{w\in W}]$ is also stable in $[U, V, W, (\succ_h)_{h\in H}]$. Unfortunately, this is not necessarily true. Indeed, our Theorem 1 ensures that there is a problem $[U, V, W, (\succ_h)_{h\in H}]$ with no weakly stable matchings and where linear orders $(\succ_w)_{w\in W}$ are strongly UV-lexicographic. Hence, if $(\succ_{U,w}, \succ_{V,w})$ are the linear orders underlying $\succ_w$, then it is impossible for a stable matching of $[U, V, W, (\succ_h)_{h\in U\cup V}, (\succ_{U,w}, \succ_{V,w})_{w\in W}]$ to be stable in $[U, V, W, (\succ_h)_{h\in H}]$. Furthermore, as the following example illustrates, a stable matching of a three-sided problem $[U, V, W, (\succ_h)_{h\in U\cup V}, (\succ_{U,w}, \succ_{V,w})_{w\in W}]$ can be unstable in the associated problem with mixed preferences $[U, V, W, (\succ_h)_{h\in H}]$, even when the latter has stable outcomes.

Example 2 Consider a three-sided matching problem with mixed preferences $[U, V, W, (\succ_h)_{h\in H}]$ where the sets of agents are given by $U = \{u_1, u_2, u_3\}$, $V = \{v_1, v_2, v_3\}$, and $W = \{w_1, w_2, w_3\}$. Suppose that strict linear orders $(\succ_h)_{h\in U\cup V}$ are restricted to satisfy the following conditions

$\succ_{u_1}$	$\succ_{u_2}$	$\succ_{u_3}$	$\succ_{v_1}$	$\succ_{v_2}$	$\succ_{v_3}$
v_1	v_2	v_3	w_1	w_2	w_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Moreover, each agent $w \in W$ has strongly UV-lexicographic preferences $\succ_w$ represented by strict linear orders $\succ_{U,w}$ and $\succ_{V,w}$, which are restricted to satisfy

$\succ_{U,w_1}$	$\succ_{V,w_1}$	$\succ_{V,w_2}$	$\succ_{V,w_3}$
u_1	v_2	v_1	v_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$

CYC instance with no weakly stable matchings and 20 agents on each market side (cf., Lerner and Lerner (2022)).

[4] Zhang and Zhong (2021, Theorem 2) shown that any problem $[U, V, W, (\succ_h)_{h\in U\cup V}, (\succ_{U,w}, \succ_{V,w})_{w\in W}]$ has a stable matching.

We affirm that the matching $M = \{(u_1, v_1, w_2), (u_2, v_2, w_1), (u_3, v_3, w_3)\}$ is stable in the problem $[U, V, W, (\succ_h)_{h\in U\cup V}, (\succ_{U,w}, \succ_{V,w})_{w\in W}]$. Indeed, since all agents in U are paired with their best alternative in V, if M is blocked by (u, v, w) then $(u, v) \in \{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$. Also, since all agents in W are paired with their best alternative in V, if M is blocked by (u, v, w) then (v, w) belongs to $\{(v_1, w_2), (v_2, w_1), (v_3, w_3)\}$. Hence, M has no blocking triplet.

However, in $[U, V, W, (\succ_h)_{h\in H}]$ the matching M is blocked by the triplet (u_1, v_1, w_1). Indeed, the agent u_1 continues to be paired with v_1, the agent v_1 improves her well-being as $w_1 \succ_{v_1} w_2$, and the agent w_1 improves her situation as $(u_1, v_1) \succ_{w_1} (u_2, v_2)$, where the last property is a consequence of the fact that $\succ_{w_1}$ is UV-lexicographic.

Notice that, for any preference profile compatible with the restriction above, the matching characterized by $\{(u_1, v_1, w_1), (u_2, v_2, w_2), (u_3, v_3, w_3)\}$ is stable in $[U, V, W, (\succ_h)_{h\in H}]$.[5]

5 Concluding remarks

In a recent article, Lam and Plaxton (2021) disproved the conjecture of Eriksson et al. (2006) about the existence of weakly stable matchings in three-sided problems with cyclic preferences. In this work, we use the result of Lam and Plaxton (2021) to guarantee that there are three-sided matching problems with mixed preferences with no stable solutions. Essentially, we obtain our result by reducing the existence of weakly stable matchings in a problem with cyclic preferences to the existence of weakly stable matchings in a problem with mixed preferences in which agents in W have UV-lexicographic preferences.

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[5] Let $[U, V, W, (\succ_h)_{h\in U\cup V}, (\succ_{U,w}, \succ_{V,w})_{w\in W}]$ be a three-sided matching problem with separable preferences and $[U, V, W, (\succ_h)_{h\in H}]$ be the associated three-sided problem with mixed preferences, in which every linear order $\succ_w$ is strongly UV-lexicographic and represented by $(\succ_{U,w}, \succ_{V,w})$. Given a matching $\overline{M}$ that is stable in the problem $[U, V, W, (\succ_h)_{h\in U\cup V}, (\succ_{U,w}, \succ_{V,w})_{w\in W}]$ but blocked by a triplet $(\overline{u}, \overline{v}, \overline{w})$ in $[U, V, W, (\succ_h)_{h\in H}]$, Zhang and Zhong (2021) claim that $\overline{v} \succ_{V,\overline{w}} \overline{M}_V(\underline{w})$ (see the item (2) in the "necessity" part of the proof of Theorem 4 in Zhang and Zhong (2021, page 935)). Unfortunately, this is not necessarily true. Indeed, by choosing $\overline{M} = M$ and $(\overline{u}, \overline{v}, \overline{w}) = (u_1, v_1, w_1)$ in Example 2, we have that $\overline{M}_V(\underline{w}) \succ_{V,\overline{w}} \overline{v}$ because of $v_2 \succ_{V,w_1} v_1$. Essentially, since $\succ_{\overline{w}}$ is UV-lexicographic, $(\overline{u}, \overline{v}) \succ_{\underline{w}} \overline{M}(\overline{w})$ does not necessarily imply that $\overline{v} \succ_{V,\overline{w}} \overline{M}_V(\underline{w})$.

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